PMM U.S.S.R.,Vol.51,No.1,pp.63-71,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

ON THE BEHAVIOUR OF SOLUTIONS OF DYNAMIC PROBLEMS NEAR THE EDGE OF THE CONTACT DOMAIN OF ELASTIC BODIES*

I.V. SIMONOV

The asymptotic forms of the solutions of problems of linear dynamic elasticity theory in the neighbourhood of a moving point of separation of different contact and separation conditions of bodies of dissimilar nature (collision and recoil /rebound/ problems) are studied. Conditions of the type of inequalities of a kinematic, dynamic, and energetic nature are taken into account. This enables complete a priori information on the subsonic speed domains, where the solutions are singular or continuous, or not realized, to be given.

The nature of the singularity of the solutions as applied to the dynamics of cracks on an interface is investigated in /1, 2/.

1. Preliminary considerations. Let us consider the domains $\Omega_1, \Omega_2 \in R^3$ filled with homogeneous linearly elastic bodies 1 and 2, $S = \bar{\Omega}_1 \cap \bar{\Omega}_2$ is the common boundary (surface), $\Gamma(t)$ is the edge of the contact domain, Q is a point belonging to a regular piece of both the curve Γ and the surface S, where the velocities $\mathbf{c}_1$ and $\mathbf{c}_2$ of the point Q with respect to bodies 1 and 2 and the acceleration $(d/dt)\,\mathbf{c}_j$ are bounded functions of t, and $\mathbf{n}$ is the normal to Γ at the point Q and a vector tangent to S.

Physically, the kinematics mentioned describe the collision of elastic bodies with smooth surfaces at small angles, and possibly, for a large discontinuity in the tangential velocities. The velocities of the particles $\mathbf{u}^j$ due to strains are considered to be small compared with the wave velocities and are always calculated in reference systems frozen in media 1 and 2 and do not change under a coordinate transformation. The reason for fixing the vectors $\mathbf{u}^j$ is the non-invariance of the equations of linear elasticity theory under a Galileo transformation; d/dt and $\partial/\partial t$ do not differ.

Localization in initial problems, correct in formulations, for the mentioned system of bodies - the passage to a Cartesian coordinate system connected with the point Q and a non-uniform stretching of the coordinates /2-4/, results in (canonically singular) limit problems. They contain a truncated separating system of Lamé motion equations to describe plane steady motions (with velocities $c_j = \mathbf{n}\mathbf{c}_j$) of the elastic half-planes in the coordinate system $x = x_1$, $y = x_2$; the x axis is directed along $\mathbf{n}$ and the y axis is perpendicular to S. Separation and imperfect contact conditions are posed for $x > 0$ and $x < 0$, $y = 0$. By virtue of the specific features of the passage to the limit (extension along the x, y axes would occur asymptotically "more rapidly" than extension along the x_3 axis), the velocity components $c_j' = \mathbf{c}_j - \mathbf{n}c_j$ tangential to the curve Γ will not occur in the limit equations of motion.

We discuss the question of the formulation of the contact conditions. The Coulomb dry friction law (below, x, y is a fixed unextended reference system)

$$\tau = -k\sigma_{22}\mathbf{v}/|\mathbf{v}| \Rightarrow \sigma_{m2} = -k_m\sigma_{22} \tag{1.1}$$

and the linear law of viscous resistance to shear

$$\tau = \eta\mathbf{v} \Rightarrow \sigma_{m2} = \eta v_m,\quad m = 1,\ 3 \tag{1.2}$$

$$\tau = (\sigma_{12}, \sigma_{32}),\ \mathbf{v} = (v_1, v_3),\ v_m = [u_m] + v_m^\circ,\ (v_1^\circ, v_3^\circ) = \mathbf{c}_2 - \mathbf{c}_1$$

will be mainly considered.

Here σ_{ml}^j, u_m^j are stress tensor and mass flow rate vector components, where the superscrip on the functions defines the medium (sometimes removed), $\mathbf{v}$ is the total slip velocity, $k > 0$ and $\eta > 0$ are coefficients of friction, and the square brackets denote jumps in the functions on transferring from the upper to the lower edge.

The viscous friction conditions are uncoupling. This is not so for (1.1): the functions describing plane and antiplane deformation are related by non-linear dependences, which complicates the problem. The main purpose of this paper is to obtain information on the singularities of the solutions, principally about the highest terms of the asymptotic expansions, since the influence of neglected parts in the equations of motion can appear in subsequent orders. Taking the above into account, we make a simplifying assumption about the constancy of the coefficients of friction in directions

*Prikl.Matem.Mekhan.,51,1,84-94,1987

$$k_m = kv_m / |\mathbf{v}| = \text{const}, \quad m = 1, 3 \tag{1.3}$$

We note that condition (1.3) is satisfied exactly if the initial formulation corresponds to plane strain ($k_3 = 0, k_1 = k \operatorname{sgn} v_1$) or plane shear ($k_1 = 0, \ k_3 = k \operatorname{sgn} v_3$). In other cases (1.3) means that the vector $\mathbf{v}$ does not change direction on approaching the curve Γ and a posteriori verification is needed.

2. Formulation of the problem (plane strain). The desired functions u_m^j and σ_{ml}^j on the interface $y = 0$ of the elastic half-planes satisfy conditions of no stresses for $x > 0$

1) $$\sigma_{m2}^j = 0, \ j, m = 1, 2$$

and an additional condition of an asymptotic nature

$$[u_2/c] = u_2^1/c_1 - u_2^2/c_2 = -d\delta/dx \leqslant 0, \quad 0 < x < x_*, \quad x_* \to 0 \tag{2.1}$$

(δ is the distance between edges) preventing "overlapping" of the edges near the contact point Q; one of the following conditions for $x < 0$

2) slip with dry friction

$$[u_2] = [\sigma_{22}] = 0, \ \sigma_{12}^j = -k_1\sigma_{22}^j, \ k_1 v_1 > 0$$

3) slip with viscous friction

$$[u_2] = [\sigma_{22}] = 0, \ \sigma_{12}^j = \eta v_1, \ v_1 \neq 0$$

4) a condition of the "comb" type (roll over without contraction)

$$[u_1] = [\sigma_{12}^1 = 0, \ \sigma_{22}^j = 0 \ (\text{ here } \quad c_1 = c_2, \ v_1^\circ = 0)$$

and additional conditions in the form of the inequalities

$$\sigma_{22}^j (x, 0) \leqslant 0, \quad x < 0; \ 0 \leqslant F < \infty \tag{2.2}$$

denoting no attraction forces between the surfaces and the extremity and non-negativity of the energy flux F at the point $x = y = 0$ (discussed in /2, 5, 6/).

We use the complex representations due to Galin /7/ in a somewhat modified form which is convenient for considering plane stationary problems of the theory of elasticity for solving canonical singular problems. Introduction of the complex potentials $\chi_m^j (z_{lj})$ /2/ is equivalent to satisfaction of the equations in the domains $y > 0$ and $y < 0$ (c should just be replaced by c_j in relations (1.3) from /2/).

On the interface $y = 0$

$$\begin{aligned} &\sigma_{12}^j = \operatorname{Im} \chi_1^j, \quad u_1^j = c_j \operatorname{Re} \{b_{2j}\chi_1^j + a_j\chi_2^j\} \\ &\sigma_{22}^j = \operatorname{Re} \chi_2^j, \quad u_2^j = -c_j \operatorname{Im} \{a_j\chi_1^j + b_{1j}\chi_2^j\} \\ &a_j = \frac{\beta_{1j}\beta_{2j} - \beta_j}{2\mu_j R_j}, \quad b_{mj} = \frac{\beta_{mj}(1-\beta_j)}{2\mu_j R_j}, \quad \beta_{mj} = \sqrt{1 - \frac{c_j^2}{c_{mj}^2}} \\ &z_{mj} = x + i\beta_{mj} y, \ \beta_j = {}^1/_2 (1 + \beta_{2j}^2), \ R_j = \beta_{1j}\beta_{2j} - \beta_j^2 \end{aligned} \tag{2.3}$$

Here c_{1j}, c_{2j} are the velocities of volume expansion and shear waves, μ_j are shear moduli, R_j are Rayleigh functions (c_{Rj} are single positive roots of the Rayleigh equations $R_j(c) = 0$), and $m, j = 1, 2$.

The estimates

$$|\chi_m^j| < \text{const} \cdot |z|^{-1/2}, \quad z = z_{lj} \to 0, \quad j, m, l = 1, 2 \tag{2.4}$$

result from the condition of finiteness of the flux F.

The equalities $[\sigma_{12}] = [\sigma_{22}] = 0$, that hold for all the conditions 1)-4) will be satisfied if the relation between the functions χ_m^j are determined by analytic continuation relationships

$$\chi_1^1(z) = -\overline{\chi_1^2(\bar z)} \equiv \chi_1(z), \quad \chi_2^1(z) = \overline{\chi_2^2(\bar z)} \equiv \chi_2(z) \tag{2.5}$$

The converse is true /2/. Taking account of (2.3) and (2.5) we have

$$\begin{aligned} &[u_1] = c \operatorname{Re} \{q\chi_1 + d\chi_2\}, [[u_2] = -c\operatorname{Im} \{d\chi_1 + p\chi_2\} \\ &d_1 = a_1 - c_0 a_2, \quad p = b_{11} + c_0 b_{12}, \ q = b_{21} + c_0 b_{22} \\ &c = c_1, \ c_0 = c_2/c_1 = 1 + v_1^\circ/c \end{aligned} \tag{2.6}$$

The velocities c_1 and c_2 are of the order of the wave velocities (otherwise, quasistatic). Consequently, in practice $v_1^\circ/c \ll 1$, the velocities of motion of the elastic bodies themselves are, as a rule, small compared with the wave velocities. The exception is ricochet when the velocity of relative motion v_1° can be commensurate with the velocities c_{2j} and a linear approximation still remains true. The approximation $c_0 \approx 1$ is considered in conditions 2), 3) and below.

For the sequel the analysis of the zeros of the Rayleigh functions $p(c), q(c), s(c) = d^2 - pq$

as well as the function $d(\mathbf{c})$ $(\mathbf{c}=(c_1, c_2))$ is important. For $c_1 = c_2$ $(v_1^\circ = 0)$ this analysis and the exposition of the physical meaning of the zeros, the velocities of a different kind of Rayleigh boundary waves, can be found in /2, 8, 9/. For $v_1^\circ \neq 0$ it is best to perform it for specific realizations because of the remarks made, as well as because of the large number of parameters and possible versions. The zeros of the functions mentioned and the solutions of the equation $c_1c_2 = 0$ are pricked out of the subsonic velocity domain under consideration $|c_j| < c_{2j}$. We denote the set of velocities obtained by C° (the remaining parameters are fixed). Sometimes degenerate situations are examined individually.

3. Mathematical method of solution. The problems examined below for the use of complex representations and (2.5) reduce to a Riemann-Hilbert boundary value problem: find the vector function $\chi(z)=(\chi_1, \chi_2)$ holomorphic in the upper z half-plane and continuously continuable everywhere on the real axis with the exception of the point $z = 0$ by means of the boundary condition

$$\operatorname{Im}\{D\chi\} = 0,\ y = +0,\ 0 < |x| < \infty \tag{3.1}$$

where D is a piecewise-constant non-degenerate matrix taking the values D_0, D_1 for $x < 0, x > 0$, the constraint (2.4) at the singular point $z = 0$, and the additional conditions resulting from (2.1)-(2.3), (2.6) (not made specific here). The distinction from the formulation of generalized Riemann-Hilbert problems /10/ is in the absence of the requirement of finiteness of the order at infinity.

We note the general approach to the solution of problem (3.1) on the basis of investigating a generalized Riemann-Hilbert boundary value problem /10, 11/. The problem often uncouples at once into a chain of two scalar problems for the components, each of which is reduced by linear substitution to the form

$$\operatorname{Im}\chi = G_1(x),\ x > 0;\ a\operatorname{Re}\chi - b\operatorname{Im}\chi = G_2(x),\ x < 0 \tag{3.2}$$

where $a, b \in R^1$, G_1, G_2 are Holder continuous real functions in their domains of definition; for one of the components there will certainly be $G_1 = G_2 \equiv 0$. The general solution of (3.2) can be represented in the form of the sum of a particular solution of the inhomogeneous problem χ°, which is easily selected below, as a rule, and a general solution of the corresponding homogeneous problem, the product of the canonical solution z^ρ /11/ by the power series

$$\chi = z^\rho P_N(z) + \chi^\circ(z),\ P_N = N_0 + N_1 z + \ldots \tag{3.3}$$

$\rho = \pi^{-1}\operatorname{arctg}(a/b),\ b \neq 0;\ \rho = -{}^1/_2,\ b = 0;\ N_n \in R^1.$

We always draw a slit for uniformization of the ambiguous functions encountered along the half-axis $y = 0, x \leqslant 0$, and we select the branch according to the condition $-\pi \leqslant \arg z \leqslant \pi$.

If the problem is not split at once, we reduce it to a Hilbert problem /10/. We introduce the piecewise-holomorphic vector $\mathbf{Y}(z)$ with the jump line $y = 0$ according to the rule

$$\mathbf{Y}(z) = D_1\chi,\ y > 0;\quad \mathbf{Y}(z) = \overline{\mathbf{Y}(\bar z)},\quad y < 0 \tag{3.4}$$

We formulate the conjugate problem for the vector $\mathbf{Y}(z)$

$$\mathbf{Y}^+ = \mathbf{Y}^-,\quad x > 0;\quad \mathbf{Y}^+ = A\mathbf{Y}^-, x < 0;\quad A = D_1 D_0^{-1}\bar D_0 \bar D_1^{-1} \tag{3.5}$$

$$\mathbf{Y}(z) = \overline{\mathbf{Y}(\bar z)},\quad y < 0;\quad |\mathbf{Y}| < \text{const}\cdot|z|^{-1/2},\quad z \to 0 \tag{3.6}$$

where the plus or minus superscripts denote contraction on the $y = 0$ axis from above or below. We find the eigenvalues of the matrix A, following /10, 12/,

$$\det(A - \lambda E) = 0 \tag{3.7}$$

In the case under consideration we have two roots of (3.7), $\lambda_1\lambda_2 \neq 0$, the singular points are regular by virtue of (2.4) /12/. If the roots are prime $(\lambda_1 \neq \lambda_2)$, then two linearly independent solution exist for (3.6) /12/

$$W_m = z^{\rho_m}\sum_{n=0}^{\infty} N_n^{(m)} z^n,\quad m = 1, 2,\quad N_n^{(m)} \in C,\quad \rho_m = \frac{\ln\lambda_m}{2\pi i} \tag{3.8}$$

where the superscripts ρ_m are selected uniquely from the interval $-{}^1/_2 \leqslant \operatorname{Re}\rho_m < {}^1/_2$, i.e., $\ln\lambda_m$ is the branch of $\operatorname{Ln}\lambda_m$ for $-\pi \leqslant \arg\lambda_m \leqslant \pi$.

If $\lambda_1 = \lambda_2$, then the linearly independent solutions have a more complex structure /12/

$$W_1 = z^\rho\sum_{n=0}^{\infty} N_n^{(1)} z^n,\quad W_2 = z^\rho\sum_{n=0}^{\infty} N_n^{(2)} z^n + M\ln z\ W_1(z) \tag{3.9}$$
$$\rho = (\ln\lambda_1)/(2\pi i),\ M,\ N_n^{(m)} \in C$$

The solution $\chi(z)$ is expressed in terms of the vector $\mathbf{W} = (W_1, W_2)$ by means of the linear transformation

$$\chi = B\mathbf{W} \tag{3.10}$$

Definite constraints, which it is desirable to find, are imposed on the coefficients of the non-degenerate matrix $B = \{B_{lm}\}$. To do this we allow the possibility of diagonalization of the matrix by using the matrix T /13/

$$T^{-1}AT = \Lambda = \begin{Vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{Vmatrix}, \quad T = \begin{Vmatrix} 1 & 1 \\ t_1 & t_2 \end{Vmatrix}$$

The substitution $\mathbf{Y} = T\mathbf{W}$ in (3.5) results in a split conjugate problem for the vector $\mathbf{W}$

$$\mathbf{W}^+ = \mathbf{W}^-, \ x > 0; \ \mathbf{W}^+ = \Lambda\mathbf{W}^-, \ x < 0 \tag{3.11}$$

whose solution is defined by (3.8) for $\lambda_1 \neq \lambda_2$ while the solution of the original problem (3.1) is now written in the form

$$\chi = D_1^{-1}(Y_1, Y_2), \ Y_1 = W_1 + W_2, \ Y_2 = t_1W_1 + t_2W_2$$

Conditions resulting from the first requirement of (3.6): $T\mathbf{W}(z) = \overline{T\mathbf{W}(\bar{z})}$ are imposed on $N_n^{(m)}$. Thus, if the numbers ρ_m are real (and distinct), then this equality is equivalent to the condition $W_m(z) = \overline{W_m(\bar{z})}$, which means

$$N_n^{(m)} \in R^1, \ n = 0, 1, 2, \ldots, \ m = 1, 2$$

If the numbers ρ_m are complex, then

$$\rho_1 = \bar{\rho}_2, \ t_1 = \bar{t}_2, \ z^{\rho_1} = \overline{z^{\rho_2}}$$

$$\mathbf{W}(z) = T^{-1}\overline{T}\,\overline{\mathbf{W}(\bar{z})} = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix} \overline{\mathbf{W}(\bar{z})} \Rightarrow W_m(z) = \overline{W_l(\bar{z})}$$

and the necessary and sufficient constraints on the coefficients $N_n^{(m)}$ are

$$N_n^{(l)} = \overline{N_n^{(m)}}; \ n = 0, 1, 2, \ldots; \ m, l = 1, 2; \ m \neq l$$

The transformation $\mathbf{Y} = T\mathbf{W}$ becomes degenerate in the case of multiple roots, and problem (3.11) generates just one linear independent solution. As is shown in /12/, the other has the form (3.9), and the relation between the coefficients B_{lm} and the constraints on the coefficients M, $N_n^{(m)}$ will be sought by direct subsitution of (3.10) into (3.1).

The solutions (3.8), (3.9) form a subgroup similar to that requiring homogeneity of the boundary conditions. The functions $b_1W_m(b_2z)$, $b_1, b_2 \in R^1, m = 1, 2$ are also solutions.

Results without intermediate calculations and, as a rule, without reference to this section are announced below.

4. Slip with dry friction-separation. We formulate the boundary value problem for the vector $\chi(z)$, an adequate physical problem for two elastic half-planes under contact conditions with slip and dry friction in the domain $x < 0$, $y = 0$ and no stresses for $x > 0$, $y = 0$

$$\begin{aligned} &\operatorname{Im} \chi_1 = \operatorname{Re} \chi_2 = 0, \ p \operatorname{Im} \chi_2 \geqslant 0, \ x > 0 \\ &\operatorname{Im} \chi_1 = -k_1 \operatorname{Re} \chi_2, \ \operatorname{Im} \{d\chi_1 + p\chi_2\} = 0, \ k_1v_1 > 0, \ x < 0 \end{aligned} \tag{4.1}$$

where condition (2.1) is taken into account. We also take account of conditions (2.2) and (2.4).

This problem separates into a homogeneous problem for the component χ_2 and an inhomogeneous problem for the subsequent determination of the function χ_1 whose solutions are

$$\begin{aligned} &\chi_1 = P_M(z) + k_1z^\rho P_N(z), \ \chi_2 = iz^\rho P_N(z), \ y \geqslant 0 \\ &\rho = \pi^{-1} \operatorname{arctg} [-p/(k_1d)], \ d \neq 0 \\ &[u_1] \sim cqk_1x^{\rho+m}N_m + cqM_0, \ [u_2] \sim -cpx^{\rho+m}N_m, \ x > 0 \\ &\sigma_{12} = -k_1\sigma_{22}, \ \sigma_{22} \sim (-1)^{m+1} \sin(\pi\rho) \ | x |^{\rho+m} N_m \\ &[u_1] \sim (-1)^m cq_0k_1^{-1} \cos(\pi\rho) \ | x |^{\rho+m}N_m + cq\,M_0, \ x < 0 \\ &F = 0, \ q_0 = k_1^2q + p \end{aligned} \tag{4.2}$$

It is assumed here that the coefficients N_0, N_1 can vanish. We select the number m for confirmation of the inequalities from (4.1) and (2.2) under the condition that m is the least of the possible integers. We always determine the signs of the functions from the sign of the highest term of the asymptotic form as $|x| \to 0$. Let us formulate the results of this analysis. The domain of existence of the singular solution ($\rho < 0$, $dpk_1 > 0$, $N_0 < 0$, $m = 0$) is the set of points $\mathbf{c} \in C_*$ given by the formula

$$p < 0 \cap \{cq_0 < 0 \cup [q_0 = 0 \cap dp\,(cq\,M_0 + v_1^\circ) > 0\} \tag{4.3}$$

This is the domain of super-Rayleigh subsonic collision velocities. In the remaining cases the solution is continuous at the point $z = 0$ and the number m is selected according to the conditions $(v_1 \approx v_1^\circ + cqM_0 \neq 0)$

$$dv_1 < 0 \cap p \geqslant 0 \Rightarrow m = 0,\ \rho > 0,\ N_0 > 0 \tag{4.4}$$
$$dv_1 < 0 \cap p < 0 \Rightarrow m = 2,\ \rho < 0,\ N_2 < 0$$
$$dv_1 > 0 \Rightarrow m = 1$$

from which the solution (4.3) should be eliminated. The systems of inequalities (4.3) and (4.4) are not contradictory and their solutions do not intersect, which indicates the solvability of the initial formulations of the problems. The version $N_0 = N_1 = 0$ is realized for super-Rayleigh subsonic recoil velocities if $c_1 \approx c_2 < 0$.

We will now analyse limit situations. The case $q = 0$ is in no way remarkable. For $p = 0$ $\rho = 0$ and the solution is regular at the point $z = 0$. In the important special case $d = 0$ (for instance, identical materials and $c_1 = c_2$) the changes in (4.2)-(4.4) are:

$$\rho = -{}^1/_2,\ F = -{}^1/_2\, \pi cpN_0^2 \geqslant 0,\ pN_m \geqslant 0,\quad (-1)^m N_m < 0 \Rightarrow \tag{4.5}$$
$$p \leqslant 0 \ \cap cp < 0 \Rightarrow m = 0,\ N_0 < 0$$
$$p > 0 \Rightarrow m = 1,\ N_1 > 0,\ N_0 = 0$$
$$p < 0 \cap cp > 0 \Rightarrow m = 2,\ \ N_2 < 0,\ \ N_0 = N_1 = 0$$

(regarding the calculation of the flux F for $\rho = -{}^1/_2$ see /3, 14/; $F = 0$ for $\rho > -{}^1/_2$).

In the other limit case $k_1 = 0$ (no friction)

$$\chi_1 = P_M(z),\ \chi_2 = iz^{-1/2}P_N(z) \tag{4.6}$$

and the system of inequalities from (4.5) governing the selection of the number m, the number for the beginning of the series $P_N(z)$, remains in force.

5. Slip with viscous friction - separation. The boundary conditions of the Riemann-Hilbert problem are formulated as follows $(y = 0)$:

$$\operatorname{Im}\chi_1 = \operatorname{Re}\chi_2 = 0,\ p^\circ \operatorname{Im}\chi_2 \geqslant 0,\ x > 0 \tag{5.1}$$
$$\operatorname{Im}\chi_1 = \eta c \operatorname{Re}\{q\chi_1 + d\chi_2\} + \eta v_1^\circ,$$
$$\operatorname{Im}\{d\chi_1 + p\chi_2\} = 0,\ x < 0$$

and conditions (2.2) and (2.4) should be appended. We find the particular solution in the form

$$\chi_1^\circ = -(cq)^{-1}v_1^\circ,\quad \chi_2^\circ \equiv 0,\quad q \neq 0;\quad \chi_1^\circ = \pi^{-1}\eta v_1^\circ \ln z,\ \chi_2^\circ = ip^{-1}d\eta v_1^\circ,\quad q = 0;$$

Following Sect.3, we introduce the vector $\mathbf{Y} = (\chi_1, i\chi_2)$ for which we obtain problem (3.5), (3.6) where

$$D_0 = \begin{Vmatrix} \gamma & -2\eta c\, dp \\ 2\eta c\, dq & -\bar{\gamma} \end{Vmatrix} \frac{1}{\Delta},\quad \gamma = p + i\eta c s_0$$
$$\Delta = p + i\eta cs,\ s = d^2 - pq,\ s_0 = d^2 + pq$$

We calculate the roots of (3.7) and the coefficients diagonalizing the matrix T

$$\lambda_{1,2} = \Delta^{-1}(i\eta cs_0 \pm \sqrt{b}),\ b = p^2 + \eta^2c^2(s^2 - s_0^2) \tag{5.2}$$
$$\sqrt{b} > 0,\quad b > 0;\ \sqrt{b} = i\sqrt{|b|},\ b < 0$$
$$t_{1,2} = (2\eta\, cdp)^{-1}(p \mp \sqrt{b})$$

The function $b(\mathbf{c})$ plays an important part in the subsequent analysis. Passage through this function denotes a qualitative passage from solutions with monotonic singularities to solutions with singularities of the oscillating type. It can hence be seen that zeros of this function exist at least for $c_1 = c_2$ including even in the pre-Rayleigh velocity range. Because of the large number of possible versions, we do not perform a detailed analysis of the zeros of the function b but we denote the set of points $\mathbf{c} \in R^2$ where $b > 0$ by C_+ (the remaining parameters are fixed) and the set of points where $b < 0$ by C_- while C_0 is the set of zeros of the function b.

Let $\mathbf{c} \in C_+$. Then we have the case of prime roots λ_m and real superscripts ρ_m. As follows from (3.8) and (5.2)

$$|\lambda_m| = 1,\quad -{}^1/_2 < \rho_m < {}^1/_2,\quad t_1t_2 = \frac{q}{p},\quad \rho_{1,2} = \frac{1}{\pi}\operatorname{arctg}\frac{p \mp \sqrt{b}}{2\eta cd^2}; \tag{5.3}$$
$$t_1, t_2 \in R^1$$
$$pq > 0 \cap cp < 0 \Rightarrow -{}^1/_2 < \rho_m < 0,\quad pq > 0 \cap cp > 0 \Rightarrow 0 < \rho_m < {}^1/_2,$$
$$m = 1, 2,\ pq < 0 \Rightarrow -{}^1/_2 < \rho_1 \operatorname{sgn} c \leqslant 0,\ 0 < \rho_2 \operatorname{sgn} c < {}^1/_2$$

The mention made above regarding the signs of ρ_m are still inadequate to make a judgment on the singularity domains of the solutions. This is finally clarified in verifying the inequalities. The general solution of problem (5.1) has the form

$$\chi_1 = W_1 + W_2 + \chi_1^\circ, \ i\chi_2 = t_1 W_1 + t_2 W_2 \tag{5.4}$$

where the functions W_m are defined by (3.8). The deduction that the solution undergoes an infinite discontinuity at the point $z = 0$ (at least one of the numbers ρ_m is negative and $N_0^{(m)} \neq 0,\ m = 1, 2)$ in the set

$$p < 0 \cap \{(pq > 0 \cap cp < 0) \cup pq < 0\} \cap c \in C_+$$

and is continuous in the remaining domain C_+ in which $N_0^{(m)} \neq 0$ if $p \geqslant 0 \cap \rho_m > 0$ and $N_0^{(m)} = 0$ if $(p > 0 \cap \rho_m < 0) \cup (p < 0 \cap \rho_m > 0)$ follows from conditions (2.1) and (2.2) $(F = 0)$ and (5.3), (5.4).

According to the remarks about the intensity coefficients, the stress and velocity asymptotic forms on the interface can be written down by using (2.3), (2.6), (5.4) as well as in the domain $|z| \ll 1$.

Zeros of the function q belong to the set C_+ We will present the solution in this degenerate case

$$\chi_1 = -\eta c d z^\rho P_N(z) + \eta v_1^\circ \pi^{-1} \ln z, \quad \chi_2 = i z^\rho P_N(z) + \chi_2^\circ, \quad N_n \in R^1,$$
$$\rho = \pi^{-1} \operatorname{arctg}(p/(\eta c d))$$

Its analysis duplicates the analysis of the solution in Sect.4. It is curious that for $q \neq 0$ the functions $[u_1] = -v_1^\circ$, $\sigma_{11}^j = \mathrm{const}\ (j)$, $\sigma_{m2}^j \equiv 0$, $j, m = 1, 2$, correspond to the particular solution $\chi_1^\circ = -(cq)^{-1} v_1^\circ$, $\chi_2^\circ = 0$, i.e., it compensates the given slip velocity v_1°.

In the domain C_- we have

$$\lambda_1 \bar{\lambda}_2 = 1, \ t_1 = \bar{t}_2, \ |t_1| = \sqrt{q/p}, \ qp > 0 \tag{5.5}$$
$$\rho = \rho_1 = \rho' + i\rho'', \ \rho_2 = \bar{\rho}; \ 2\pi\rho' = \arg \lambda_1$$
$$-\pi < \arg \lambda_1 < \pi; \ 2\pi\rho'' = \ln |\lambda_2|$$

and the solution $\chi(z)$ is defined by (3.8), (5.4). The expansion $\delta(x)$ is sign-variable for $x > 0$, a consequence of oscillating type singularities. There is no success in satisfying conditions of the inequality type in C_-. Therefore, the formulation of problem (5.1), meaning also the corresponding initial problems, contains a defect, and such a combination of conjugate conditions of elastic half-planes is not realized if $c \in C_-$.

The question of eliminating the defect remains open but we note that the oscillation zone of the solution occupies the domain

$$|z| < 1 \cap |\rho'' \ln |z|| \geqslant \pi \Rightarrow |z| < \exp(-2\pi^2/\ln|\lambda_1|) = l_0$$
$$|\lambda_1| = \beta_0 + \sqrt{\beta_0^2 - 1}, \ \beta_0 = |\Delta|^{-1} \eta c s_0 > 1$$

It is seen that for the values $|\lambda_1| \geqslant 1$ the characteristic dimension l_0 of this zone is extremely small (it cannot be taken into account). Then an intermediate asymptotic form exists for $l_0 \ll |z| \ll L$, where L is the outer large characteristic dimension. In this domain the solution is described approximately by (5.4), (5.5), where we should set $\rho'' = 0$, i.e., we have a monotonic asymptotic form, for which confirmation of the additional conditions (2.1), (2.2) already has meaning. In sum, we obtain $N_0 \neq 0$; $p > 0 \Rightarrow \rho' > 0$; $p < 0 \Rightarrow \rho' < 0$. However, there are velocities for which $\ln|\lambda_1| \gg 1$. This is the neighbourhood of the points $c_j = C_{Rj} \in C_-$ ($|\lambda_1| \to \infty$ as $|c_j| \to C_{Rj}$). The oscillation zone of the solution grows for near-Rayleigh velocities and the solution found loses meaning. Analogous behaviour of the solution is remarked in an analysis of the combination separation-complete contact /1, 9/.

The measure of the domain C_- equals zero but the solution does not acquire an oscillating nature for $\eta = 0$ (no friction) or $d = 0$. It can be said that the appearance of fluctuations is related to the presence of two physical factors, viscous friction and inhomogeneity of the materials.

Let $c \in C_0$, $\lambda_1 = \lambda_2$. For $p \neq 0$

$$p = 4\eta^2 c^2 q d^2, \ q \neq 0, \ d \neq 0, \ t_1 = t_2 = (2\eta c d)^{-1}$$
$$s = d^2(1 - \eta_0^2), \ s_0 = d^2(1 + \eta_0^2), \ \eta_0 = 2\eta c q \neq 0$$
$$\rho = \pi^{-1} \operatorname{arctg} \eta_0, \ \operatorname{sgn} \rho = \operatorname{sgn}(cq), \ p = 0 \Rightarrow \lambda_1 = \lambda_2 = 1, \ \rho = 0$$

The linearly independent solutions W_m are defined in (3.9), where $M, N_n^{(m)} \in R^1$. We determine the connection between the coefficients $B_{ml} \in R^1$ and M by substituting (3.10) into (5.1) and equating coefficients of terms of identical power. Omitting the details of the tedious analysis, we present the result. Two conditions should be imposed on the five numbers B_{ml} and M

$$B_{22} = t_1 B_{12}, \ B_{21} = t_1 B_{11} + \pi t_1 (\eta_0 + \eta_0^{-1}) M B_{12}$$

and since W_1 and W_2 already contain an arbitrary factor, we can set $B_{11} = B_{12} = 1$. Therefore

$$\chi_1 = W_1 + W_2 - v_1{}^\circ/(cq),\ i\chi_2 = t_2 W_1 + t_1 W_2,\ t_2 = t_1 + \pi t_1 (\eta_0 + \eta_0^{-1}) M \tag{5.6}$$

It can be verified that (5.6) satisfies all the initial boundary conditions of equality type. Satisfying inequalities (2.1) and (2.2), we arrive at the deduction

$$p \leqslant 0 \cap cq < 0 \Rightarrow \rho < 0, \quad N_0^{(m)} \neq 0, \quad pcq < 0 \Rightarrow N_0^{(m)} = 0, \quad N_1^{(m)} \neq 0$$

$$p \geqslant 0 \cap cq > 0 \Rightarrow \rho > 0, \quad N_0{}^{(m)} \neq 0, \ m = 1, 2$$

Here $F = 0$ because $\rho > -{}^1/_2$.

We present the solution for the case $d = 0$ ($\mathbf{c} \in C^\circ$)

$$\chi_1 = z^\rho P_M(z) + \chi_1{}^\circ, \quad \chi_2 = iz^{-1/2} P_N(z), \quad M_n, N_n \in R^1$$

$$\rho = \pi^{-1} \operatorname{arctg}(\eta cq), \ \operatorname{sgn} \rho = \operatorname{sgn}(cq)$$

By confirmation of conditions (2.1) and (2.2) we establish

$$d = 0 \cap cp < 0 \cap p < 0 \Rightarrow N_0 < 0, \ F = -{}^1/_2 \pi N_0{}^2\, cp \tag{5.7}$$

For the remaining values of the zeros of the function $d(\mathbf{c}): N_0 = 0, F = 0$.

6. Slip with constant shear resistance-separation. In the previous versions the friction forces sometimes grew to infinity as one approached the point Q. It is natural to impose a constraint on these friction forces (as is repeatedly mentioned in the literature), for instance, by setting $\sigma_{12} = \tau_1 = \text{const}, \tau_1 v_1 > 0$ in the contact zone. Such a condition can be interpreted as the presence of a thin strip of plastic flow (for example, /15/). The corresponding problem for the function $\chi(z)$ uncouples at once; its solution is produced below

$$\chi_1 = P_M(z) + \pi^{-1}\tau_1 \ln z, \ \chi_2 = iz^{-1/2} P_N(z) + id\tau_1/p$$

$$M_n, N_n \in R^1, \ p \neq 0$$

For $p = 0$ the conditions of the problem are contradictory ($\tau_1 \neq 0$). By confirming the inequality and the sign-matching condition for τ_1 and v_1, we arrive at the following deductions

$$\mathbf{c} \in C_1 \Rightarrow N_0 < 0, \ \ F = -{}^1/_2\, \pi N_0{}^2\ cp; \ \ C_1 = \{\mathbf{c} : p \leqslant 0 \cap cp < 0\} \tag{6.1}$$

where $d \neq 0 \Rightarrow cd\,\tau_1 < 0, d = 0 \Rightarrow cq < 0$

$$\mathbf{c} \overline{\in} C_1 \Rightarrow F = N_0 = 0, \ N_1 > 0; \quad \text{where} \quad d \neq 0 \Rightarrow pd\tau_1 p < 0; \ d = 0 \Rightarrow p > 0; \ q \neq 0 \Rightarrow cq < 0; \ q = 0 \Rightarrow \tau_1 v_1 > 0 \tag{6.2}$$

Analysis of the cases $N_0 = N_1 = 0, N_2 \neq 0, \ldots$ does not alter conditions (6.2) in principle. Consequently, we conclude that the collision **regime** being studied is realized for not all subsonic values of the velocity $\mathbf{c}$. Under conditions (6.1), the solution is singular, under conditions (6.2) it contains $\ln z$ but satisfaction of the condition $cq \leqslant 0$ is required in addition to $\mathbf{c}$ "not belonging" to the domain mentioned in (6.1). However, it is important to clarify whether there are intersections of the domains of non-realization of the contact conditions $\sigma_{12} = \tau_1$ with the singularity domains in the dry and viscous friction cases. It can be shown strictly that these intersections are empty sets. Hence the deduction: when it is necessary to restrict friction, a scheme with a thin plastic strip is suitable; when this scheme is unsuitable (it is impossible to introduce plasticity outside the contact point Q), the plasticity condition can be replaced by one of the friction conditions and a physically meaningful result can be obtained (with the stipulations of Sect.5) with a smooth growth of the stresses on the contact area. In other words, the problem of determining the singularities with alternative conditions on the contact area (friction, the plasticity condition) is solvable.

7. Separation-condition of the "comb" type. This case models the adhesion of gear transmissions, say, with fine and particular teeth. The boundary value problem (3.1) with condition 1)-4) is split, and the answer is

$$\chi_1 = z^{-1/2} P_N(z), \ \chi_2 = iP_M(z), \ \ N_n, M_n \in R^1$$

$$cq \geqslant 0 \Rightarrow N_0 \neq 0, F = {}^1/_2\, \pi cq\, N_0{}^2 \geqslant 0; cq < 0 \Rightarrow N_0 = F = 0$$

The stresses and velocities are singular, particularly in the velocity range $0 < c < c_{Rj}$, $j = 1,2$ ("engagement of the gear teeth" occurs). In the opposite case $0 < -c < c_{Rj}$ the desired functions are continuous.

8. Formulation of the plane shear problems. For the plane shear equations (w is the displacement in the direction of the x_3 axis)

$$\beta^2{}_{2j} w^j_{,xx} + w^j_{,yy} = 0, \ \sigma_{13}{}^j = \mu_j w_{,x}{}^j$$
$$\sigma_{23}{}^j = \mu_j w_{,y}{}^j, \ u_3{}^j = -c_j w_{,x}{}^j \tag{8.1}$$

we consider the following boundary conditions $(y = 0,\ j = 1, 2)$.

1°. No stresses (separation or slip without friction)

$$\sigma_{23}{}^j = 0$$

2°. Slip with dry friction

$$\sigma_{23}{}^j = -k_3\sigma_{22}{}^j,\quad \sigma_{23}{}^j v_3 > 0,\quad [\sigma_{22}] = 0$$

3°. Slip with viscous friction

$$\sigma_{23}{}^j = \eta v_3 \neq 0 \quad (v_3 = v_3{}^\circ + [u_3])$$

In all cases the inequalities (2.2) are additional conditions. We introduce the piecewise-holomorphic function $\Phi(z_{2j})$ by means of the formulas

$$\Phi = c_j{}^{-1}\mu_j\beta_{2j}u_3{}^j + i\sigma_{23}{}^j;\ \Phi(z) = -\overline{\Phi(\bar z)},\ y > 0 \tag{8.2}$$

$$\sigma_{23}{}^j = \operatorname{Im}\Phi(z_{2j}),\quad [u_3] = \beta\operatorname{Re}\Phi(x + i0),\quad \beta = \frac{c_1}{\mu_1\beta_{21}} + \frac{c_2}{\mu_2\beta_{22}}$$

which is equiavlent to satisfying the equation of motion in the domain and the condition $[\sigma_{23}] = 0$ on the interface.

9. Slip with dry friction - separation. We substitute the expression for σ_{22} from solution (4.2) in the boundary conditions to determine the function $\Phi(z)$ corresponding to the physical conditions 1° and 2°

$$\operatorname{Im}\Phi = 0,\ x > 0;\ \operatorname{Im}\Phi = -k_3\sigma_{22}(x, 0),\ v_3\operatorname{Im}\Phi > 0,\ x < 0\ (y = 0).$$

The function

$$\Phi = k_3 z^\rho P_N(z) + P_L(z),\ L_n \in R^1,\ \operatorname{Im} z > 0 \tag{9.1}$$

is the complete solution of the problem.

We will write down the asymptotic form $(y = 0)$

$$\sigma_{23} = 0,\ [u_3] \sim \beta k_3 x^{\rho+m}N_m + \beta L_0,\ x > 0$$

$$\sigma_{23} \sim (-1)^m k_3 \sin(\pi\rho)\,|x|^{\rho+m}N_m,$$

$$[u_3] \sim (-1)^m \beta k_3 \cos(\pi\rho)\,|x|^{\rho+m}N_m + \beta L_0$$

where the number m is still selected according to the rules (4.3) and (4.4). From the condition for matching the signs of v_3 and k_3 in the case $\rho < 0,\ m = 0,\ N_0 < 0$ when the sign of v_3 determines the component $\sim |x|^\rho$, the inequality $\beta < 0$ follows. It is not completely in agreement with (4.3). Therefore, the combined singular solution is realized in a velocity subset, shrunken as compared with (4.3), namely at the intersection

$$C_2 = \{\mathbf{c} : \beta < 0 \cap C_*\} \tag{9.2}$$

The *a priori* conditions (1.3) are here satisfied asymptotically. If $c_1 = c_2 = c$, then (9.2) simplifies

$$C_2 = \{\mathbf{c} : c < 0 \cap p < 0 \cap [(qk_1{}^2 + p) > 0 \cup (qk_1{}^2 = -p \cap dv_1 < 0)]\} \tag{9.3}$$

and it is seen that the measure of the subset C_2 is substantially less than the measure of the set C_* ($\operatorname{mes} C_2 \to 0$ as $k_1 \to 0$). For $\mathbf{c} \overline{\in} C_2$ it must be assumed that $m + \rho > 0$, and the number m is determined by expanded rules (4.4). Then the solutions (4.2) and (9.1) are not singular but

$$k_l = kv_l / |\mathbf{v}|,\ v_1 \approx cqM_0 + v_1{}^\circ,\ v_3 \approx \beta L_0 + v_3{}^\circ,\ l = 1, 3$$

i.e., conditions (1.3) that are taken by assumption are asymptotically exact, where the error is $O(|x|^{m+\rho}) + O(x),\ x \to -0$.

For $k = 0$ solution (9.1) is regular for all $\mathbf{c} \in C^\circ$, while solution (4.6) is singular in the domain (4.3), where k_1 should be set equal to zero. Every connection between these solutions is lost but it is shown that the degeneration $k \to 0$ is sometimes irregular.

Let us make general deductions about the case of contact with dry friction. In C° there exist velocity ranges of the point Q forming a subset C_2, where the solutions (9.1) and (4.2) are singular and do not contradict all the assumptions made. These solutions are continuous in the subset $C^\circ \setminus C_2$ and, moreover, have velocities at which they possess smoothness at the singular point according to (4.4). Judging by (9.2) and (9.3), the measure of the singularity domain of the solutions is small compared with $\operatorname{mes} C^\circ$ and lies in the super-Rayleigh velocity range.

10. Slip with viscous friction - separation. Satisfying the boundary conditions

$$\operatorname{Im}\Phi = 0,\ x > 0;\ \operatorname{Im}\Phi = \eta\beta\operatorname{Re}\Phi + \eta v_3{}^\circ,\ x < 0$$

by the usual scheme, we obtain

$$\Phi = z^\rho P_L(z) - v_3{}^\circ\beta^{-1},\ -{}^1\!/_2 < \rho = \operatorname{arctg}(\eta\beta) < {}^1\!/_2,\ L_n \in R^1$$

The velocity field $u_3{}^j(x, y)$, analogous to plane strain (Sect.5), acquires constants compensating the given slip velocity of bodies as rigid bodies. The solution is singular for $\beta < 0$ and continuous for $\beta > 0$.

REFERENCES

1. ACHENBACH J.D., BAZANT Z.P. and KHETAN R.P., Elastodynamic neartip fields for a rapidly propagating interface crack. Intern. J. Engng. Sci., 14, 9, 1976.
2. SIMONOV I.V., On the subsonic motion of the edge of a shear displacement with friction along the interface of elastic materials. PMM, 47, 3, 1983.
3. KOSTROV B.V., NIKITIN L.V. and FLITMAN L.M., Mechanics of brittle fracture. Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, 3, 1969.
4. CHERPANOV G.P., Mechanics of Brittle Fracture. Nauka, Moscow, 1974.
5. FREUND L.B. and CLIFTON R.J., On the uniqueness of plane elastodynamic solutions for running cracks, J. Elast., 4, 4, 1974.
6. FREUND L.B., The mechanics of dynamic shear crack propagation, J. Geophys. Res., Ser.B, 84, 5, 1979.
7. GALIN L.A., Contact Problems of Elasticity Theory, Gostekhizdat, Moscow, 1953.
8. GRINCHENKO V.T. and MELESHKO V.V., Harmonic Oscillations and Waves in Elastic Bodies. Naukova Dumka, Kiev, 1981.
9. GOL'DSHTEIN R.V., On stationary crack propagation on the rectilinear boundary connecting two elastic materials. Inzh. Zh., Mekhan., Tverd. Tela, 5, 1966.
10. VEKUA N.P., Systems of Singular Integral Equations and Certain Boundary Value Problems. Nauka, Moscow, 1970.
11. MUSKHELISHVILI N.I., Singular Integral Equations. Nauka, Moscow, 1968.
12. SMIRNOV V.I., Course in Higher Mathematics, 3, pt.2, Nauka, Moscow, 1974.
13. LANCASTER P., Theory of Matrices. Nauka, Moscow, 1978.
14. ACHENBACH J.D., Wave propagation, singular elastodynamic stresses and fracture. Theoretical and Applied Mechanics, Trans. 14th Internat. Congress IUTAM, Mir, Moscow, 1979.
15. KOSTROV B.V., Mechanics of the Focus of a Tectonic Earthquake. Nauka, Moscow, 1975.

Translated by M.D.F.

PMM U.S.S.R.,Vol.51,No.1,pp.71-76,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

PRESSURE OF A STAMP OF ALMOST ANNULAR PLANFORM ON AN ELASTIC HALF-SPACE*

A.B. KOVURA and V.I. SAMARSKII

The generalization of the problem of the impression of an annular stamp without friction into an elastic half-space /1, 2/ is considered. The contact domain has an axis of symmetry and is a ring bounded by curves of almost circular shape. The half-space material is isotropic and homogeneous. Determination of the pressure under the stamp reduces to finding two functions of a complex variable, analytic in a circle, by means of boundary conditions of mixed type. The unknown constants on the right-hand sides of the boundary conditions are determined under the assumption that the dimensions of the holes in the stamp are small. The results from /3, 4/, referring to the case of annular or almost circular stamps, are essentially used here.

1. A stamp with a flat base, whose side surface is formed by cylinders $r = r_1(\varphi)$ and $r = r_2(\varphi)\,(r_2(\varphi) < r_1(\varphi),\ \varphi \in [-\pi, \pi])$ is impressed without friction in an elastic half-space $z \geqslant 0$. Outside the stamp the surface of the half-space is force-free. For a given settling of the stamp w_0 determine the pressure $p(r, \varphi)$ in the contact domain S, a non-circular ring $r_2^2(\varphi) < r^2 < r_1^2(\varphi)$.

Following /5/, the potential theory problem that occurs here for the half-space $z > 0$ can be written in the form

*Prikl.Matem.Mekhan.,51,1,95-100,1987